Proceedings of the Jangjeon Mathematical Society
21 (2018), No. 2. pp. 167 – 170

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2018.21.2.167

GENERAL STRUCTURE OF TWO-DIMENSIONAL QUASIREPRESENTATIONS OF GROUPS

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Abstract. The structure of two-dimensional quasirepresentations of groups and of the corresponding pseudorepresentations is described.

§ 1. Introduction

Recall that a mapping π of a given group G into the family of operators on a Banach space E is said to be a *quasirepresentation* of G if

$$\|\pi(g_1 g_2) - \pi(g_1) - \pi(g_2)| \le \varepsilon, \qquad g_1, g_2 \in G,$$

for some ε, which is usually assumed to be sufficiently small and is referred to as a *defect* of π and a quasirepresentation of G is said to be a *pseudorepresentation* of G if $\pi(g^n)$ is similar to $\pi(g)^n$, $n \in \mathbb{N}$, with the help of an operator sufficiently close to the identity operator; see [1–3].

There are results concerning low-dimensional quasirepresentations. The one-dimensional quasirepresentations of groups with small defect were described in [4]. A rather special problem concerning the structure of two-dimensional real triangular quasirepresentations of groups were treated by Faiziev, my former postgraduate student [5].

General results of [1–4] enable us to describe completely the structure of two-dimensional quasirepresentations of groups.

2010 *Mathematics Subject Classification.* Primary 22A99, Secondary 22A25.

Key words and phrases. quasirepresentation of a group, pseudorepresentation, pure pseudorepresentation.

Partially supported by the Russian Foundation for Basic Research under grant no. 18-01-00398.

§ 2. Main results

Theorem. *Let G be a group and let π be a quasirepresentation of G with a sufficiently small defect ($\varepsilon < 1/3$) in a two-dimensional normed space E. Then π belongs to one of the following classes of quasirepresentations:*

(1) *π is an unbounded ordinary irreducible representation of G;*

(2) *π is a bounded ordinary irreducible representation of G;*

(3) *π is a bounded irreducible quasirepresentation of G which is not a representation;*

(4) *π is a completely reducible mapping whose diagonal components are small perturbations of bounded pure one-dimensional pseudorepresentations (recall that a pseudorepresentation is said to be* pure *if its restriction to every amenable subgroup is an ordinary representation of the subgroup) that are not ordinary representations;*

(5) *π is a small perturbation of a completely reducible mapping whose diagonal components are small perturbations of a one-dimensional ordinary representation and a bounded one-dimensional pure pseudorepresentation which is not an ordinary representation;*

(6) *π is a small perturbation of a completely reducible mapping whose diagonal components are small perturbations of one-dimensional ordinary representations one of which is bounded;*

(7) *π is a small perturbation of a completely reducible ordinary bounded representation;*

(8) *π is a reducible not completely reducible unbounded triangular ordinary representation;*

(9) *π is a reducible not completely reducible unbounded triangular quasirepresentation whose diagonal components are ordinary bounded one-dimensional representations.*

Proof. The main tool to prove the above classification is the general theorem on the structure of an arbitrary finite-dimensional quasirepresentation of an arbitrary group (see, e.g., [1]). According to this theorem, if T is a quasirepresentation of G in a finite-dimensional vector space E_T, E_T^* is the conjugate space of E_T, L is the set of vectors $\xi \in E_T$ for which the orbit $\{T(g)\xi,\, g \in G\}$ is bounded, M is the set of functionals $f \in E_T^*$ for which the orbit $\{T(g)^* f,\, g \in G\}$ is bounded in E_T^* (L and the annihilator $M^\perp$ are in fact vector subspaces in E_T invariant under T), then, in the collection of subspaces $\{0\}$, $L \cap M^\perp$, $M^\perp$, $L + M^\perp$, and $E = E_T$ (in ascending order), the matrix $t(g)$ of the operator $T(g)$, $g \in G$, in block form (with respect to the decomposition of the space E into the direct sum of subspaces $L \cap M^{\perp}$, $M^{\perp} \setminus (L \cap M^{\perp})$, $L \setminus (L \cap M^{\perp})$, and $E \setminus (L + M^{\perp}))$, where "\" stands for a complementary subspace, is

$$t(g) = \begin{pmatrix} \alpha(g) & \varphi(g) & \sigma(g) & \tau(g) \\ 0 & \beta(g) & 0 & \rho(g) \\ 0 & 0 & \gamma(g) & \chi(g) \\ 0 & 0 & 0 & \delta(g) \end{pmatrix}, \qquad g \in G. \tag{1}$$

(Here we have $t_{23}(g) = 0$, because L is invariant under T.) Moreover,

(i) the mappings α, δ, γ, σ, and χ are bounded;

(ii) the mappings t_1 and t_2 defined by the equations

$$t_1(g) = \begin{pmatrix} \alpha(g) & \varphi(g) \\ 0 & \beta(g) \end{pmatrix}, \qquad t_2(g) = \begin{pmatrix} \beta(g) & \rho(g) \\ 0 & \delta(g) \end{pmatrix},$$

are representations of the group G;

(iii) the mapping τ is a quasicocycle with respect to the representations t_1 and t_2, i.e., the mapping

$$(g, h) \to \tau(gh) - \alpha(g)\tau(h) - \varphi(g)\rho(h) - \tau(g)\delta(h), \qquad g, h \in G,$$

is bounded.

The irreducibility condition in cases (1)–(3) means that the representation space belongs to one of the four diagonal blocks, which gives the answer immediately. The unboundedness condition implies that the quasirepresentation in question is an ordinary representation (see, e.g., [6]), which explains the structure of the assertion in (8), the absence of unboundedness condition in (4) and (7), and a "partial" participation of unbounded mappings in (5) and (6). The logically remaining possibility (9) is, in particular, related to the nontrivial quasicharacters (say, for identity diagonal representations). This completes the proof.

§ 3. Question

It is still not known whether or not every two-dimensional pseudorepresentation of a group is automatically pure.

Acknowledgments

I thank Professor Taekyun Kim for the invitation to publish this paper in the Proceedings of the Jangjeon Mathematical Society.

References

1. A. I. Shtern, *Quasisymmetry. I*, Russ. J. Math. Phys. **2** (1994), no. 3, 353–382.
2. A. I. Shtern, *Finite-dimensional quasi-representations of connected Lie groups and Mishchenko's conjecture*, J. Math. Sci. **159** (2009), no. 5, 653–751.
3. A. I. Shtern, *Remarks on finite-dimensional locally bounded finally precontinuous quasirepresentations of locally compact groups*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), no. 4, 469–480.
4. A. I. Shtern, *A version of van der Waerden's theorem and a proof of Mishchenko's conjecture on homomorphisms of locally compact groups*, Izv. Math. **72** (2008), no. 1, 169–205.
5. V. A. Faiziev, *Two-dimensional real triangular quasirepresentations of groups*, Fundam. Prikl. Mat. **1** (1995), no. 4, 1129–1132.
6. A. I. Shtern, *Exponential stability of quasihomomorphisms into Banach spaces and a Ger-Šemrl theorem*, Russ. J. Math. Phys. **22** (2015), no. 1, 141–142.

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